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LETTERS TO THE EDITORS

NOTE ON "LAMINAR FREE CONVECTION ALONG A VERTICAL PLATE AT EXTREMELY SMALL GRASHOF NUMBERS"

F. J. SURIANO, K-T. YANG and J. A. DONLON, *Int. J. Heat Mass Transfer* **8**(5), 815–831 (1965).

IN [1], Suriano *et al.* have studied theoretically the laminar free convection for an isothermal vertical finite plate. Their approach was to construct an asymptotic expansion in the Grashof number with $G \to 0$ as a limit. They computed numerically the first three terms in the series and claim the results are valid for Grashof numbers between zero and one.

In this note it will be demonstrated that the zero-order problem as posed in [1] cannot be solved. This indicates that either the expansion scheme is incorrect or, at best, is a singular perturbation where some of the boundary conditions can only be satisfied by a supplemental expansion.

The difficulty was overlooked because of a pitfall which occurs in the use of a computer. Frequently, a researcher is solving a problem on an infinite domain or as time becomes infinite. The customary approach is to solve a sequence of problems on progressively larger domains or for larger times. Since the computer storage is limited, one cannot compute for all arbitrarily large domains. This problem is overcome by introducing an artificial criteria and comparing the results from runs with different domains. It sometimes occurs that the criteria may be satisfied but the proper solution for an infinite domain or time is not obtained.

The nomenclature of [1] will be used throughout the discussion. In Fig. 1 the plate occupies the x axis from $-0{\cdot}5$ to $+0{\cdot}5$ and the fluid extends to infinity in all directions. The zero-order problem for the temperature from [1] is

$$\nabla^2\theta^{(0)} = 0$$

on the plate,

$$y = 0,\ -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}$$

$$\theta^{(0)} = 1$$

and at infinity

$$\theta^{(0)} = 0.$$

Physically this represents conduction from the plate through a stagnant fluid to infinity.

A numerical computer solution was obtained in [1] and because of symmetry only a quarter plane was computed. As is customary in computer solutions the boundary condition at infinity was located at a finite position. The authors of [1] state that "the locations of infinity in both space-coordinate directions are determined by extending the grid system out in these directions until the calculated perturbation functions approach their respective values asymptotically, . . . , it has been found that locations of infinity may be represented by $x = \pm 7{\cdot}5$ and $y = 7{\cdot}0, \ldots$,". In this note an equivalent problem is solved exactly to show that the location of "infinity" affects the result no matter how far from the origin it is placed.

Elliptic coordinates are introduced as shown schematically in Fig. 1. The relations with the rectangular coordinates are:

$$\xi + i\eta = \cosh\,[2(x + iy)]$$

$$x = \tfrac{1}{2}\cosh\xi\cos\eta$$

$$y = \tfrac{1}{2}\sinh\xi\sin\eta$$

$$\frac{4x^2}{\cosh^2\xi} + \frac{4y^2}{\sinh^2\xi} = 1$$

$$\frac{4x^2}{\cos^2\eta} - \frac{4y^2}{\sin^2\eta} = 1.$$

The Laplace equation will allow a separation of variable solution in this system so we choose to place the infinite boundary condition on the ellipse given by $\xi = \xi_0$.

$$\theta^{(0)} = 0 \qquad \text{at} \quad \xi = \xi_0.$$

If the location of the boundary is actually immaterial then choosing an ellipse instead of a rectangle as used in [1] should be permissible. The boundary, condition on the plate transforms to

$$\theta^{(0)} = 1 \qquad \text{at} \quad \xi = 0.$$

The solution is then given in elliptic coordinates by

$$\theta^{(0)} = 1 - \xi/\xi_0.$$

The answer depends on the choice of ξ_0 but more importantly the limit $\xi_0 \to \infty$ gives $\theta^{(0)} = 1$. This does not satisfy the original boundary condition that $\theta^{(0)} \to 0$ at infinity. Thus the zero-order problem in [1] is not well-posed. Physically it is similar to trying to find the temperature in a rod with one end held at $\theta^{(0)} = 1$ and the other end at $\theta^{(0)} = 0$, a perfectly good problem for a finite rod but not when there is an infinite distance between the ends.

If the problem can be approached by the asymptotic series assumed in [1], it is at best a singular perturbation where two matched expansions will be required. As noted in [1], Mahony [2] investigated the zero Grashof number limit for spheres and cylinders. He encountered the same problem we have pointed out; the far boundary condition cannot be satisfied. He overcame this difficulty by patching his solution to solutions for the far wake. His results then depend upon the method and point of patching. This curve fitting procedure is not as theoretically pleasing as a true matching process used in the method of inner and outer expansions.

It is of some interest to compute the heat transfer from the plate. The temperature gradient at the plate is

$$\frac{\partial \theta^{(0)}}{\partial y} = \frac{2/\xi_0}{[1 - (2x)^2]^{\frac{1}{2}}}.$$

This is singular at the ends of the plate $x = \pm\frac{1}{2}$. However, it is integrable and the Nusselt number can be found as

$$N = \frac{hL}{k} = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \theta^{(0)}}{\partial y} = \pi/\xi_0.$$

This result depends upon the location of the infinite boundary condition as does the temperature profile.

The artificial criteria used in [1] to compare solutions gave an answer. This result was not the proper limit as shown by the exact solution. In retrospect then, how could a researcher determine whether the real answer has been found by the computer or not? For certain problems mathematicians have produced existance and uniqueness theorems; however, more often than not, this information is missing. There appears to be no definite answer. A general idea of what to expect might be obtained by studying exact solutions for related or similar problems. In the present case a clue that the computer result is not correct is found in the solution for the cylinder given by Mahony [2].

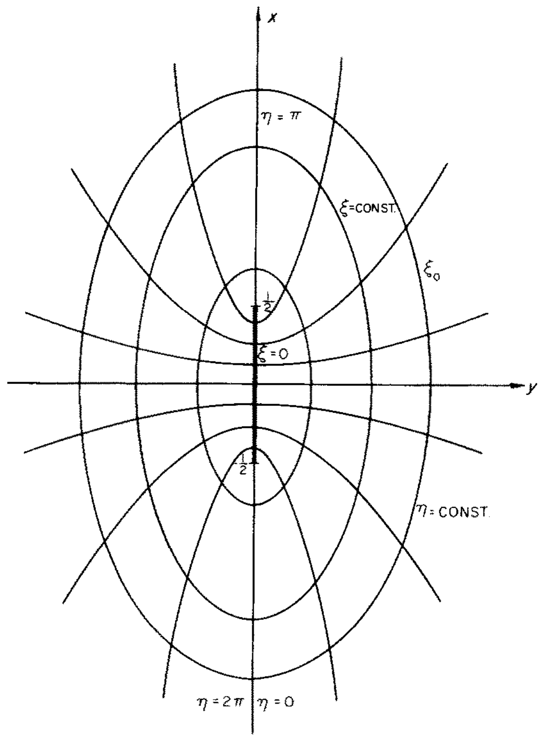

FIG. 1.

REFERENCES

1. F. J. SURIANO, K-T. YANG and J. A. DONLON, Laminar free convection along a vertical plate at extremely small Grashof numbers, *Int. J. Heat Mass Transfer* **8**, 815 (1965).
2. J. J. MAHONY, Heat transfer at small Grashof numbers, *Proc. R. Soc.* **A238**, 412 (1956–1957).

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REJOINDER

To DISCUSS our previous paper [1], it is necessary to examine both the nature of the mathematical solution and its relevance to the real physical situation. Professor Panton's remarks on the zeroth-order or the pure conduction solution are mathematically correct and well taken. However, as we intend to show below, the difficulty pointed out by him does not affect the validity of the results of our published solution.

In view of the great complexity of the governing differential equations for laminar free convection at small Grashof numbers and the general difficulty in obtaining the solutions, theoretical studies, by necessity, must be based on certain